

## INTEGRATION

## Answers

- 1 a**  $\frac{3x+5}{(x+1)(x+3)} \equiv \frac{A}{x+1} + \frac{B}{x+3}$   
 $3x+5 \equiv A(x+3) + B(x+1)$   
 $x=-1 \Rightarrow 2=2A \Rightarrow A=1$   
 $x=-3 \Rightarrow -4=-2B \Rightarrow B=2$   
 $\therefore \frac{3x+5}{(x+1)(x+3)} \equiv \frac{1}{x+1} + \frac{2}{x+3}$
- b**  $= \int \left( \frac{1}{x+1} + \frac{2}{x+3} \right) dx$   
 $= \ln|x+1| + 2 \ln|x+3| + c$
- 2**  $\frac{3}{(t-2)(t+1)} \equiv \frac{A}{t-2} + \frac{B}{t+1}$   
 $3 \equiv A(t+1) + B(t-2)$   
 $t=2 \Rightarrow 3=3A \Rightarrow A=1$   
 $t=-1 \Rightarrow 3=-3B \Rightarrow B=-1$   
 $\therefore \int \frac{3}{(t-2)(t+1)} dt$   
 $= \int \left( \frac{1}{t-2} - \frac{1}{t+1} \right) dt$   
 $= \ln|t-2| - \ln|t+1| + c$   
 $= \ln \left| \frac{t-2}{t+1} \right| + c$
- 3 a**  $\frac{6x-11}{(2x+1)(x-3)} \equiv \frac{A}{2x+1} + \frac{B}{x-3}$   
 $6x-11 \equiv A(x-3) + B(2x+1)$   
 $x=-\frac{1}{2} \Rightarrow -14=-\frac{7}{2}A \Rightarrow A=4$   
 $x=3 \Rightarrow 7=7B \Rightarrow B=1$   
 $\therefore \int \frac{6x-11}{(2x+1)(x-3)} dx$   
 $= \int \left( \frac{4}{2x+1} + \frac{1}{x-3} \right) dx$   
 $= 2 \ln|2x+1| + \ln|x-3| + c$
- b**  $\frac{14-x}{x^2+2x-8} \equiv \frac{A}{x+4} + \frac{B}{x-2}$   
 $14-x \equiv A(x-2) + B(x+4)$   
 $x=-4 \Rightarrow 18=-6A \Rightarrow A=-3$   
 $x=2 \Rightarrow 12=6B \Rightarrow B=2$   
 $\therefore \int \frac{14-x}{x^2+2x-8} dx$   
 $= \int \left( \frac{2}{x-2} - \frac{3}{x+4} \right) dx$   
 $= 2 \ln|x-2| - 3 \ln|x+4| + c$
- c**  $\frac{6}{(2+x)(1-x)} \equiv \frac{A}{2+x} + \frac{B}{1-x}$   
 $6 \equiv A(1-x) + B(2+x)$   
 $x=-2 \Rightarrow 6=3A \Rightarrow A=2$   
 $x=1 \Rightarrow 6=3B \Rightarrow B=2$   
 $\therefore \int \frac{6}{(2+x)(1-x)} dx$   
 $= \int \left( \frac{2}{2+x} + \frac{2}{1-x} \right) dx$   
 $= 2 \ln|2+x| - 2 \ln|1-x| + c$   
 $= 2 \ln \left| \frac{2+x}{1-x} \right| + c$
- d**  $\frac{x+1}{5x^2-14x+8} \equiv \frac{A}{5x-4} + \frac{B}{x-2}$   
 $x+1 \equiv A(x-2) + B(5x-4)$   
 $x=\frac{4}{5} \Rightarrow \frac{9}{5} = -\frac{6}{5}A \Rightarrow A = -\frac{3}{2}$   
 $x=2 \Rightarrow 3=6B \Rightarrow B = \frac{1}{2}$   
 $\therefore \int \frac{x+1}{5x^2-14x+8} dx$   
 $= \int \left( \frac{\frac{1}{2}}{(x-2)} - \frac{\frac{3}{2}}{5x-4} \right) dx$   
 $= \frac{1}{2} \ln|x-2| - \frac{3}{10} \ln|5x-4| + c$
- 4 a**  $x^2-6 \equiv A(x+4)(x-1) + B(x-1) + C(x+4)$   
 $x=-4 \Rightarrow 10=-5B \Rightarrow B=-2$   
 $x=1 \Rightarrow -5=5C \Rightarrow C=-1$   
 coeffs of  $x^2 \Rightarrow A=1$
- b**  $= \int \left( 1 - \frac{2}{x+4} - \frac{1}{x-1} \right) dx$   
 $= x - 2 \ln|x+4| - \ln|x-1| + c$

**5 a**  $\frac{x^2-x-4}{(x+2)(x+1)^2} \equiv \frac{A}{x+2} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$   
 $x^2-x-4 \equiv A(x+1)^2 + B(x+2)(x+1) + C(x+2)$   
 $x = -2 \Rightarrow A = 2$   
 $x = -1 \Rightarrow C = -2$   
 coeffs of  $x^2 \Rightarrow 1 = A + B \Rightarrow B = -1$   
 $\therefore \frac{x^2-x-4}{(x+2)(x+1)^2} \equiv \frac{2}{x+2} - \frac{1}{x+1} - \frac{2}{(x+1)^2}$

**b**  $= \int \left( \frac{2}{x+2} - \frac{1}{x+1} - \frac{2}{(x+1)^2} \right) dx$   
 $= 2 \ln|x+2| - \ln|x+1| + 2(x+1)^{-1} + c$

**6 a**  $\frac{3x^2-5}{x^2-1} \equiv A + \frac{B}{x+1} + \frac{C}{x-1}$   
 $3x^2-5 \equiv A(x+1)(x-1) + B(x-1) + C(x+1)$   
 $x = -1 \Rightarrow -2 = -2B \Rightarrow B = 1$   
 $x = 1 \Rightarrow -2 = 2C \Rightarrow C = -1$   
 coeffs of  $x^2 \Rightarrow A = 3$

$\therefore \int \frac{3x^2-5}{x^2-1} dx = \int \left( 3 + \frac{1}{x+1} - \frac{1}{x-1} \right) dx$   
 $= 3x + \ln|x+1| - \ln|x-1| + c = 3x + \ln \left| \frac{x+1}{x-1} \right| + c$

**b**  $\frac{x(4x+13)}{(2+x)^2(3-x)} \equiv \frac{A}{2+x} + \frac{B}{(2+x)^2} + \frac{C}{3-x}$   
 $x(4x+13) \equiv A(2+x)(3-x) + B(3-x) + C(2+x)^2$   
 $x = -2 \Rightarrow -10 = 5B \Rightarrow B = -2$   
 $x = 3 \Rightarrow 75 = 25C \Rightarrow C = 3$   
 coeffs of  $x^2 \Rightarrow 4 = -A + C \Rightarrow A = -1$   
 $\therefore \int \frac{x(4x+13)}{(2+x)^2(3-x)} dx = \int \left( \frac{3}{3-x} - \frac{1}{2+x} - \frac{2}{(2+x)^2} \right) dx$   
 $= -3 \ln|3-x| - \ln|2+x| + 2(2+x)^{-1} + c$

**c**  $\frac{x^2-x+1}{x^2-3x-10} \equiv A + \frac{B}{x-5} + \frac{C}{x+2}$   
 $x^2-x+1 \equiv A(x-5)(x+2) + B(x+2) + C(x-5)$   
 $x = 5 \Rightarrow 21 = 7B \Rightarrow B = 3$   
 $x = -2 \Rightarrow 7 = -7C \Rightarrow C = -1$   
 coeffs of  $x^2 \Rightarrow A = 1$   
 $\therefore \int \frac{x^2-x+1}{x^2-3x-10} dx = \int \left( 1 + \frac{3}{x-5} - \frac{1}{x+2} \right) dx$   
 $= x + 3 \ln|x-5| - \ln|x+2| + c$

**d**  $\frac{2-6x+5x^2}{x^2(1-2x)} \equiv \frac{A}{x} + \frac{B}{x^2} + \frac{C}{1-2x}$   
 $2-6x+5x^2 \equiv Ax(1-2x) + B(1-2x) + Cx^2$   
 $x = \frac{1}{2} \Rightarrow \frac{1}{4} = \frac{1}{4}C \Rightarrow C = 1$   
 $x = 0 \Rightarrow B = 2$   
 coeffs of  $x^2 \Rightarrow 5 = -2A + C \Rightarrow A = -2$   
 $\therefore \int \frac{2-6x+5x^2}{x^2(1-2x)} dx = \int \left( \frac{1}{1-2x} - \frac{2}{x} + \frac{2}{x^2} \right) dx$   
 $= -\frac{1}{2} \ln|1-2x| - 2 \ln|x| - 2x^{-1} + c$

- 7  $\frac{3x-5}{(x-1)(x-2)} \equiv \frac{A}{x-1} + \frac{B}{x-2}$   
 $3x-5 \equiv A(x-2) + B(x-1)$   
 $x=1 \Rightarrow -2 = -A \Rightarrow A=2$   
 $x=2 \Rightarrow 1 = B \Rightarrow B=1$   
 $\therefore \int_3^4 \frac{3x-5}{(x-1)(x-2)} dx = \int_3^4 \left( \frac{2}{x-1} + \frac{1}{x-2} \right) dx$   
 $= [2 \ln|x-1| + \ln|x-2|]_3^4$   
 $= (2 \ln 3 + \ln 2) - (2 \ln 2 + 0) = 2 \ln 3 - \ln 2$
- 8 a  $\frac{x+3}{x(x+1)} \equiv \frac{A}{x} + \frac{B}{x+1}$   
 $x+3 \equiv A(x+1) + Bx$   
 $x=0 \Rightarrow 3 = A \Rightarrow A=3$   
 $x=-1 \Rightarrow 2 = -B \Rightarrow B=-2$   
 $\therefore \int_1^3 \frac{x+3}{x(x+1)} dx = \int_1^3 \left( \frac{3}{x} - \frac{2}{x+1} \right) dx$   
 $= [3 \ln|x| - 2 \ln|x+1|]_1^3$   
 $= (3 \ln 3 - 2 \ln 4) - (0 - 2 \ln 2) = 3 \ln 3 - 2 \ln 2$
- b  $\frac{3x-2}{x^2+x-12} \equiv \frac{A}{x+4} + \frac{B}{x-3}$   
 $3x-2 \equiv A(x-3) + B(x+4)$   
 $x=-4 \Rightarrow -14 = -7A \Rightarrow A=2$   
 $x=3 \Rightarrow 7 = 7B \Rightarrow B=1$   
 $\therefore \int_0^2 \frac{3x-2}{x^2+x-12} dx = \int_0^2 \left( \frac{2}{x+4} + \frac{1}{x-3} \right) dx$   
 $= [2 \ln|x+4| + \ln|x-3|]_0^2$   
 $= (2 \ln 6 + 0) - (2 \ln 4 + \ln 3)$   
 $= 2(\ln 2 + \ln 3) - 4 \ln 2 - \ln 3 = \ln 3 - 2 \ln 2$
- c  $\frac{9}{2x^2-7x-4} \equiv \frac{A}{2x+1} + \frac{B}{x-4}$   
 $9 \equiv A(x-4) + B(2x+1)$   
 $x = -\frac{1}{2} \Rightarrow 9 = -\frac{9}{2}A \Rightarrow A = -2$   
 $x=4 \Rightarrow 9 = 9B \Rightarrow B=1$   
 $\therefore \int_1^2 \frac{9}{2x^2-7x-4} dx = \int_1^2 \left( \frac{1}{x-4} - \frac{2}{2x+1} \right) dx$   
 $= [\ln|x-4| - \ln|2x+1|]_1^2$   
 $= (\ln 2 - \ln 5) - (\ln 3 - \ln 3) = \ln 2 - \ln 5$
- d  $\frac{2x^2-7x+7}{x^2-2x-3} \equiv A + \frac{B}{x-3} + \frac{C}{x+1}$   
 $2x^2-7x+7 \equiv A(x-3)(x+1) + B(x+1) + C(x-3)$   
 $x=3 \Rightarrow 4 = 4B \Rightarrow B=1$   
 $x=-1 \Rightarrow 16 = -4C \Rightarrow C=-4$   
 coeffs of  $x^2 \Rightarrow A=2$   
 $\therefore \int_0^2 \frac{2x^2-7x+7}{x^2-2x-3} dx = \int_0^2 \left( 2 + \frac{1}{x-3} - \frac{4}{x+1} \right) dx$   
 $= [2x + \ln|x-3| - 4 \ln|x+1|]_0^2$   
 $= (4 + 0 - 4 \ln 3) - (0 + \ln 3 - 0) = 4 - 5 \ln 3$

$$\begin{aligned} \mathbf{e} \quad \frac{5x+7}{(x+1)^2(x+3)} &\equiv \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+3} \\ 5x+7 &\equiv A(x+1)(x+3) + B(x+3) + C(x+1)^2 \\ x=-1 &\Rightarrow 2=2B \Rightarrow B=1 \\ x=-3 &\Rightarrow -8=4C \Rightarrow C=-2 \\ \text{coeffs of } x^2 &\Rightarrow 0=A+C \Rightarrow A=2 \\ \therefore \int_0^1 \frac{5x+7}{(x+1)^2(x+3)} dx &= \int_0^1 \left( \frac{2}{x+1} + \frac{1}{(x+1)^2} - \frac{2}{x+3} \right) dx \\ &= [2 \ln|x+1| - (x+1)^{-1} - 2 \ln|x+3|]_0^1 \\ &= (2 \ln 2 - \frac{1}{2} - 2 \ln 4) - (0 - 1 - 2 \ln 3) \\ &= \frac{1}{2} - 2 \ln 2 + 2 \ln 3 \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad \frac{2+x}{8-2x-x^2} &\equiv \frac{A}{4+x} + \frac{B}{2-x} \\ 2+x &\equiv A(2-x) + B(4+x) \\ x=-4 &\Rightarrow -2=6A \Rightarrow A=-\frac{1}{3} \\ x=2 &\Rightarrow 4=6B \Rightarrow B=\frac{2}{3} \\ \therefore \int_{-1}^1 \frac{2+x}{8-2x-x^2} dx &= \int_{-1}^1 \left( \frac{\frac{2}{3}}{2-x} - \frac{\frac{1}{3}}{4+x} \right) dx \\ &= \left[ -\frac{2}{3} \ln|2-x| - \frac{1}{3} \ln|4+x| \right]_{-1}^1 \\ &= (0 - \frac{1}{3} \ln 5) - (-\frac{2}{3} \ln 3 - \frac{1}{3} \ln 3) \\ &= \ln 3 - \frac{1}{3} \ln 5 \end{aligned}$$

$$\begin{aligned} \mathbf{9} \quad \mathbf{a} \quad \frac{1}{x^2-a^2} &\equiv \frac{A}{x+a} + \frac{B}{x-a} \\ 1 &\equiv A(x-a) + B(x+a) \\ x=-a &\Rightarrow 1=-2aA \Rightarrow A=-\frac{1}{2a} \\ x=a &\Rightarrow 1=2aB \Rightarrow B=\frac{1}{2a} \\ \therefore \frac{1}{x^2-a^2} &\equiv \frac{1}{2a(x-a)} - \frac{1}{2a(x+a)} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \int \frac{1}{x^2-a^2} dx &= \frac{1}{2a} \int \left( \frac{1}{x-a} - \frac{1}{x+a} \right) dx \\ &= \frac{1}{2a} (\ln|x-a| - \ln|x+a|) + c \\ &= \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + c \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \int \frac{1}{a^2-x^2} dx &= -\int \frac{1}{x^2-a^2} dx \\ &= -\frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + c \\ &= \frac{1}{2a} \ln \left| \frac{x+a}{x-a} \right| + c \end{aligned}$$

$$\begin{aligned} \mathbf{10} \quad \mathbf{a} &= \left[ \frac{1}{6} \ln \left| \frac{x-3}{x+3} \right| \right]_{-1}^1 \\ &= \frac{1}{6} (\ln \frac{1}{2} - \ln 2) \\ &= -\frac{1}{3} \ln 2 \\ \mathbf{b} &= 4 \left[ \frac{1}{2} \ln \left| \frac{x+1}{x-1} \right| \right]_{-\frac{1}{2}}^{\frac{1}{2}} \\ &= 2 (\ln 3 - \ln \frac{1}{3}) \\ &= 4 \ln 3 \\ \mathbf{c} &= \frac{3}{2} \left[ \frac{1}{4} \ln \left| \frac{x-2}{x+2} \right| \right]_0^1 \\ &= \frac{3}{8} (\ln \frac{1}{3} - 0) \\ &= -\frac{3}{8} \ln 3 \end{aligned}$$